Hamiltonian analysis of General relativity with the Immirzi parameter

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Starting from a Lagrangian we perform the full constraint analysis of the Hamiltonian for General relativity in the tetrad-connection formulation for an arbitrary value of the Immirzi parameter and solve the second class constraints, presenting the theory with a Hamiltonian composed of first class constraints which are the generators of the gauge symmetries of the action. In the time gauge we then recover Barbero's formulation of gravity.

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A formulation of General relativity using real connections as the dynamical variables of the theory has been proposed by Barbero [1] and has been used since then in the loop approach to quantum gravity [2]. The reality of the theory, as opposed to the complex variables of Ashtekar [3], is achieved at the cost of the non-polynomiallity of the Hamiltonian constraint. Barbero's formulation also leads to the so-called Immirzi ambiguity [4], which from the present point of view [5] arises from the addition to the standard action for General relativity in the tetrad-connection formalism of a term which does not affect the classical equations of motion, but which may affect the quantum theory [6]. The spectrum of the volume and area operators, and consequently the entropy of black holes, seems to depend on the Immirzi parameter β (the constant that multiplies the term added to the the action that we mentioned before). Notably there is one single value of this parameter for which the conventional expression of the entropy of black holes is reproduced [7]. Ashtekar's gravity can be obtained from the complex version of the ordinary tetrad-connection action for gravity supplemented with the Immirzi term when $\beta = \pm i$ [8,9], and Barbero's gravity can be obtained in the same fashion but with no need to complexify the theory and setting β real and non-zero [5]. It seems therefore worthwhile to make an effort to understand better these theories. Here we perform the full Hamiltonian analysis prior to any gauge-fixing for an arbitrary value of the Immirzi parameter and solve all second class constraints, presenting the theory with a Hamiltonian composed of first class constraints which are the generators of the gauge symmetries of the action. In this way we are also able to confirm Holst's result [5] which has been called in question because it was obtained from a partially gauge-fixed action.

In section I we introduce the reader to the Lagrangian of the theory and to the notation used. In section II the Hamiltonian is derived and in section III we compute the constraint algebra and find the secondary constraints. It turns out that some constraints form second class pairs. This analysis is similar to what is usually done without the Immirzi parameter, except that now there are two distinct natural choices of canonical variables. In section IV the second class constraints are solved and the explicit form of the remaining first class constraints is presented. Finally in section V we fix the boost gauge freedom and show the equivalence of this theory to Barbero's formulation of gravity.

I. THE LAGRANGIAN

We use the standard tetrad field $e^{\alpha I}(x)$ and antisymmetric connection $\omega_{\alpha IJ}(x)$, both defined on each point x of a spacetime manifold. We use Greek letters for indices in the tangent space to the manifold. Capital Latin letters stand for indices in the internal space, which is endowed with a Minkowskian metric with signature $(-1, 1, 1, 1)$. When $3+1$ decompositions will be performed, timelike indices will be labeled "t" in the tangent space and "0" in the internal space, while spacelike indices will be labelled with small Latin letters starting at "a" and "i" for the tangent space and internal space respectively. When contracting indices in the internal space we shall not care about their position, whether raised or lowered, the presence of a suitable metric (Minkowskian in the whole space or Euclidean in the spacelike part of the decomposition) being understood when necessary.

The metric on the spacetime manifold is

$$g^{\alpha\beta} = e^{\alpha I} e^{\beta I} \ . \tag{1}$$

The determinant of the tetrad field and the curvature in internal space are represented respectively by

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$$e = 1/\det(e^{\alpha I}) \tag{2}$$
$$R_{\alpha\beta IJ} = \partial_{[\alpha}\omega_{\beta]IJ} + \omega_{[\alpha|IK}\omega_{|\beta]JK} \ . \tag{3}$$

The dual of an antisymmetric tensor of second order in internal space is defined by

$$^*U_{IJ} = \frac{1}{2}\epsilon_{IJKL}U_{KL} \ . \tag{4}$$

The action is a first order action constructed from the tetrads $e^{\alpha I}$ and connection $\omega_{\alpha IJ}$. It is $S = \int d^4xL$, with

$$L = \frac{1}{2}ee^{\alpha I}e^{\beta J}\left(R_{\alpha\beta IJ} - \frac{1}{\beta}{}^*R_{\alpha\beta IJ}\right) \ , \tag{5}$$

where β is the Immirzi parameter, and it can be easily shown that it indeed describes locally General relativity at the classical level regardless of the value of β. It is very convenient to introduce for antisymmetric tensors of second order in internal space the notation [1]

$$\overset{(\beta)}{U}_{IJ} = U_{IJ} - \frac{1}{\beta}{}^*U_{IJ} \ . \tag{6}$$

This is a one-to-one relation between the β-dependent ${}^{(\beta)}U_{IJ}$ and the β-independent U_{IJ}, with inverse

$$U_{IJ} = \frac{\beta^2}{1+\beta^2}\left(\overset{(\beta)}{U}_{IJ} + \frac{1}{\beta}{}^*\overset{(\beta)}{U}_{IJ}\right) \ . \tag{7}$$

One can derive the useful relations

$$\overset{(\beta)}{(U_{[I|K}V_{|J]K})} = \overset{(\beta)}{U}_{[I|K}V_{|J]K} = U_{[I|K}\overset{(\beta)}{V}_{|J]K} \tag{8}$$
$$\overset{(\beta)}{U}_{IJ}V_{IJ} = U_{IJ}\overset{(\beta)}{V}_{IJ} \ . \tag{9}$$

With this notation the Lagrangian (5) becomes

$$L = \frac{1}{2}ee^{\alpha I}e^{\beta J}\overset{(\beta)}{R}_{\alpha\beta IJ} \ . \tag{10}$$

II. THE HAMILTONIAN

We perform the 3+1 decomposition of (10),

$$L = ee^{tI}e^{aJ}\overset{(\beta)}{R}_{taIJ} + \frac{1}{2}ee^{aI}e^{bJ}\overset{(\beta)}{R}_{abIJ} \ , \tag{11}$$

and split the tetrad field into

$$N = -\frac{1}{eg^{tt}} \tag{12}$$
$$N^a = -\frac{g^{ta}}{g^{tt}} \tag{13}$$
$$\pi^{aIJ} = ee^{t[I|}e^{a|J]} \ . \tag{14}$$

The first component is the lapse and the second three are the shift. The π^{aIJ} have 18 components. Since the tetrad has got 16 independent components, there are 6 variables in excess in this decomposition. Indeed the π^{aIJ} are subject to 6 constraints,

[1] See the added note.

$$\mathcal{C}^{ab} = \frac{1}{2}\pi^{aIJ*}\pi^{bIJ} \sim 0 \tag{15}$$

It is standard to derive that the second parcel on the right hand side of (11) becomes

$$\frac{1}{2}ee^{aI}e^{bJ} \overset{(\beta)}{R}_{abIJ} = N\mathcal{H} + N^a\mathcal{H}_a \ , \tag{16}$$

with

$$\mathcal{H}_a = \frac{1}{2}\pi^{bIJ} \overset{(\beta)}{R}_{abIJ} = \frac{1}{2} \overset{(\beta)}{\pi}{}^{bIJ} R_{abIJ} \tag{17}$$

$$\mathcal{H} = \frac{1}{2}\pi^{aIK}\pi^{bJK} \overset{(\beta)}{R}_{abIJ} = \frac{1}{2} \overset{(\beta)}{\pi}{}^{aIK}\pi^{bJK} R_{abIJ} \ . \tag{18}$$

And the first parcel on the right hand side of (11) becomes

$$\frac{1}{2}\pi^{aIJ} \overset{(\beta)}{R}_{taIJ} = \frac{1}{2} \overset{(\beta)}{\pi}{}^{aIJ} R_{taIJ} = \frac{1}{2} \overset{(\beta)}{\pi}{}^{aIJ} \dot{\omega}_{aIJ} - \frac{1}{2}\Lambda_{IJ}\mathcal{G}^{IJ} \ , \tag{19}$$

where a partial integration has been performed in the second equality, and

$$\Lambda_{IJ} = -\omega_{tIJ} \tag{20}$$

$$\mathcal{G}^{IJ} = D_a \overset{(\beta)}{\pi}{}^{aIJ} \ , \tag{21}$$

In this last formula covariant differentiation is acting in the internal space only. Equation (21) reads explicitly

$$\mathcal{G}^{IJ} = \partial_a \overset{(\beta)}{\pi}{}^{aIJ} + \omega_a{}^{IK} \overset{(\beta)}{\pi}{}^{aJK} - \omega_a{}^{JK} \overset{(\beta)}{\pi}{}^{aIK} = \partial_a \overset{(\beta)}{\pi}{}^{aIJ} + \overset{(\beta)}{\omega}{}_a{}^{IK}\pi^{aJK} - \overset{(\beta)}{\omega}{}_a{}^{JK}\pi^{aIK} \ . \tag{22}$$

Putting (11), (16) and (19) together, one ends up with the Hamiltonian

$$H = N\mathcal{H} + N^a\mathcal{H}_a + \frac{1}{2}\Lambda_{IJ}\mathcal{G}^{IJ} + \frac{1}{2}c_{ab}\mathcal{C}^{ab} \ , \tag{23}$$

where (15) arose as primary constraints resulting from the very definition (14) of the variables π^{aIJ}. The symplectic form is

$$\frac{1}{2} \overset{(\beta)}{\pi}{}^{aIJ}\dot{\omega}_{aIJ} = \frac{1}{2}\pi^{aIJ} \dot{\overset{(\beta)}{\omega}}_{aIJ} \ . \tag{24}$$

There are therefore two sets of variables which are canonically conjugate and that can be naturally chosen to parameterize phase space, β-dependent connections together with β-independent momenta, or vice-versa,

$$\left\{ \overset{(\beta)}{\omega}_{aIJ}(x), \pi^{bKL}(y) \right\} = \left\{ \omega_{aIJ}(x), \overset{(\beta)}{\pi}{}^{bKL}(y) \right\} = \delta_a^b(\delta_I^K\delta_J^L - \delta_I^L\delta_J^K)\delta^3(x-y) \ . \tag{25}$$

Both connections are $SO(3,1)$ connections.

Thus our system is described by any of the canonical pairs (25) and the totally constrained Hamiltonian (23), where ω_{tIJ}, N^a and N play the role of Lagrange multipliers for respectively the constraints (21), (17) and (18), known as the gauge, vector and scalar constraints. When performing the constraint analysis one should use the inverting equations between β-dependent and β-independent quantities, equations (6)-(7), in order to write the constraints solely in terms of canonical variables, whatever the set one chooses to use. In order to facilitate the use of β-dependent connections, we derive

$$\overset{(\beta)}{R}_{abIJ} = \partial_{[\alpha} \overset{(\beta)}{\omega}_{\beta]IJ} + \frac{\beta^2}{1+\beta^2}\left[\overset{(\beta)}{\omega}_{[\alpha|IK} \overset{(\beta)}{\omega}_{|\beta]JK} + \frac{1}{\beta}{}^{*}\left(\overset{(\beta)}{\omega}_{[\alpha|IK} \overset{(\beta)}{\omega}_{|\beta]JK} \right) \right] \ . \tag{26}$$

III. CONSTRAINT ANALYSIS

In this section we perform the constraint analysis of the Hamiltonian derived in the previous section. The resulting algebra is independent of the set of canonical pairs chosen to perform the calculations.

It is easy to check that the constraints $\mathcal{G}^{IJ}$ are the generators of internal gauge transformation, and that the combinations

$$\tilde{\mathcal{H}}_a = \mathcal{H}_a - \frac{1}{2}\omega_{aIJ}\mathcal{G}^{IJ} = \frac{1}{2}\left[\pi^{bIJ}\partial_a \overset{(\beta)}{\omega}{}_{bIJ} - \partial_b\left(\overset{(\beta)}{\omega}{}_{aIJ}\,\pi^{bIJ}\right)\right] = \frac{1}{2}\left[\overset{(\beta)}{\pi}{}^{bIJ}\partial_a\omega_{bIJ} - \partial_b\left(\omega_{aIJ}\overset{(\beta)}{\pi}{}^{bIJ}\right)\right] \tag{27}$$

are the generators of spatial diffeomorphisms. It is therefore to be expected that the Poisson bracket of any constraint with $\mathcal{G}^{IJ}$ or $\mathcal{H}_a$ vanishes on the constraint surface. In fact

$$\left\{\frac{1}{2}\mathcal{G}^{IJ}\left[\Lambda_{IJ}\right], \frac{1}{2}\mathcal{G}^{KL}\left[\Omega_{KL}\right]\right\} = \mathcal{G}^{IJ}\left[\Lambda_{IK}\Omega_{JK}\right] \tag{28}$$

$$\left\{\frac{1}{2}\mathcal{G}^{IJ}\left[\Lambda_{IJ}\right], \mathcal{H}_a\left[N^a\right]\right\} = 0 \tag{29}$$

$$\left\{\frac{1}{2}\mathcal{G}^{IJ}\left[\Lambda_{IJ}\right], \mathcal{H}\left[N\right]\right\} = 0 \tag{30}$$

$$\left\{\frac{1}{2}\mathcal{G}^{IJ}\left[\Lambda_{IJ}\right], \frac{1}{2}\mathcal{C}^{ab}\left[c_{ab}\right]\right\} = 0 \tag{31}$$

$$\left\{\mathcal{H}_a\left[M^a\right], \mathcal{H}_b\left[N^b\right]\right\} = \mathcal{H}_a\left[M^b\partial_b N^a - N^b\partial_b M^a\right] - \frac{1}{2}\mathcal{G}^{IJ}\left[M^a N^b R_{abIJ}\right] \tag{32}$$

$$\left\{\mathcal{H}_a\left[M^a\right], \mathcal{H}\left[N\right]\right\} = \mathcal{H}\left[M^a\partial_a N - N\partial_a M^a\right] + \mathcal{G}^{IJ}\left[N M^a \pi^{bIK} R_{abJK}\right] \tag{33}$$

$$\left\{\mathcal{H}_a\left[N^a\right], \frac{1}{2}\mathcal{C}^{bc}\left[c_{bc}\right]\right\} = \tfrac{1}{2}\,\mathcal{C}^{ab}\left[N^c\partial_c c_{ab} + 2c_{ac}\partial_b N^c - c_{ab}\partial_c N^c\right] + \frac{\beta^2}{2(1+\beta^2)}\mathcal{G}^{IJ}\left[c_{ab}N^a\left({}^*\pi^{bIJ} - \frac{1}{\beta}\pi^{bIJ}\right)\right] \tag{34}$$

The remaining Poisson brackets are

$$\left\{\mathcal{H}\left[M\right], \mathcal{H}\left[N\right]\right\} = -\frac{1}{2}\mathcal{H}_a\left[\left(M\partial_b N - N\partial_b M\right)\pi^{aIJ}\pi^b{}_{IJ}\right] + \frac{1}{2}\mathcal{C}^{ab}\left[\left(M\partial_a N - N\partial_a M\right){}^*\pi^{cIJ}\overset{(\beta)}{R}{}_{cbIJ}\right] \tag{35}$$

$$\left\{\frac{1}{2}\mathcal{C}^{ab}\left[c_{ab}\right], \frac{1}{2}\mathcal{C}^{cd}\left[d_{cd}\right]\right\} = 0 \tag{36}$$

$$\left\{\mathcal{H}\left[N\right], \frac{1}{2}\mathcal{C}^{ab}\left[c_{ab}\right]\right\} = \frac{1}{2}\mathcal{D}^{ab}\left[N c_{ab}\right] \;, \tag{37}$$

where

$$\mathcal{D}^{ab} = {}^*\pi^{cIJ}\left(\pi^{aIK}D_c\pi^{bJK} + \pi^{bIK}D_c\pi^{aJK}\right) \;. \tag{38}$$

We note that we obtained the characteristic Poisson bracket of the scalar constraint in General relativity, since

$$-\frac{1}{2}\pi^{aIJ}\pi^b{}_{IJ} = g(g^{tt}g^{ab} - g^{ta}g^{tb}) = qq^{ab} \;, \tag{39}$$

where q^{ab} is the 3-dimensional matrix inverse to the spatial part of the metric, $q_{ab} = g_{ab}$, and

$$q = \det(q_{ab}) = \sqrt{\det\left(-\frac{1}{2}\pi^{aIJ}\pi^b{}_{IJ}\right)} \;. \tag{40}$$

Due to (37) the constraint algebra does not close. We shall not repeat the full analysis here, which is analogous to the standard treatment of the Hilbert-Palatini action for General relativity [10]. It results that the conditions $\mathcal{D}^{ab} \sim 0$ must be imposed as secondary constraints, and it is easy to check that they form second class pairs with the constraints $\mathcal{C}^{ab}$. Their Poisson bracket is

$$\left\{\frac{1}{2}\mathcal{C}^{ab}\left[c_{ab}\right], \frac{1}{2}\mathcal{D}^{cd}\left[d_{cd}\right]\right\} =$$

$$= \mathcal{C}^{ab}\left[(c_{ab}d_{cd} - c_{ac}d_{bd})\mathcal{C}^{cd} + \beta^{-1}(c_{ab}d_{cd} + c_{bd}d_{ab} - 2c_{ac}d_{bd})qq^{cd}\right] + q^2 q^{ab}q^{cd}(c_{ac}d_{bd} - c_{ab}d_{cd}) \;. \tag{41}$$

Thus, there are no tertiary constraints, and the full Hamiltonian is

$$H = N\mathcal{H} + N^a\mathcal{H}_a + \frac{1}{2}\Lambda_{IJ}\mathcal{G}^{IJ} + \frac{1}{2}c_{ab}\mathcal{C}^{ab} + \frac{1}{2}d_{ab}\mathcal{D}^{ab} \ . \tag{42}$$

IV. SOLVING THE SECOND CLASS CONSTRAINTS

Now one must solve both $\mathcal{C}^{ab}$ and $\mathcal{D}^{ab}$, and insert their solutions in the Hamiltonian, where only the scalar, vector and gauge constraints survive. The solution to $\mathcal{C}^{ab}$ is

$$\pi^{a0i} = E^{ai} \tag{43}$$

$$\pi^{aij} = E^{a[i}\chi^{j]} \ , \tag{44}$$

where

$$\chi^i = -\frac{e^{ti}}{e^{tt}} \ . \tag{45}$$

This is a convenient way of expressing the solution to $\mathcal{C}^{ab}$, which enables us to set the time gauge in the simple form $\chi^i = 0$. The spatial metric (39) is given by

$$qq^{ab} = E^{ai}\eta_{ij}E^{bj} \tag{46}$$

with

$$\eta_{ij} = (1-\chi^k\chi^k)\delta_{ij} + \chi^i\chi^j \ . \tag{47}$$

When $\chi^i\chi^i \neq 1$ this is a positive definite metric with the property that it does not distinguish between contravariant and covariant indices of χ^i, $\eta_{ij}\chi^j = \chi^i$. The fields E^{ai} are therefore not triads but rather they bring the metric to this form (they become triads after gauge fixing and $\eta_{ij} \to \delta_{ij}$).

Inserting the solution (43)-(44) into the symplectic form (24) projects out 12 components of the connections,

$$\frac{1}{2}\pi^{aIJ}\dot{\overset{(\beta)}{\omega}}_{aIJ} = E^{ai}\dot{A}_{ai} + \zeta_i\dot{\chi}^i \ , \tag{48}$$

with

$$A_{ai} = \overset{(\beta)}{\omega}_{a0i} + \overset{(\beta)}{\omega}_{aij}\,\chi^j \tag{49}$$

$$\zeta_i = \overset{(\beta)}{\omega}_{aij}\,E^{aj} \ . \tag{50}$$

We are now working with the new set of canonical variables (A_{ai}, E^{ai}) and (χ^i, ζ_i) with non-trivial Poisson brackets

$$\{A_{ai}(x), E^{bj}(y)\} = \delta_i^j\delta_a^b\delta^3(x-y) \tag{51}$$

$$\{\chi^i(x), \zeta_i(y)\} = \delta_i^j\delta^3(x-y) \ . \tag{52}$$

The connections can be written in terms of the new variables as

$$\overset{(\beta)}{\omega}_{a0i} = A_{ai} - \overset{(\beta)}{\omega}_{aij}\,\chi^j \tag{53}$$

$$\overset{(\beta)}{\omega}_{aij} = \frac{1}{2}\left(\frac{1}{1-\chi^k\chi^k}\epsilon_{ijk}E_{al}M^{kl} - E_{a[i}\zeta_{j]}\right) \ , \tag{54}$$

where M^{ij} is symmetric and represents the components of the connections which do not show up in the symplectic form. Since the Poisson brackets of the vector and gauge constraints with $\mathcal{C}^{ab}$ vanishes on the surface of primary constraints, one expects them to be straightforwardly written in terms of the new canonical variables. Indeed

$$\mathcal{H}_a = E^{bi}\partial_{[a}A_{b]i} + \zeta_i\partial_a\chi^i + \frac{\beta^2}{1+\beta^2}\left[-E^{b[i}\chi^{j]}A_{ai}A_{bj} - A_{ai}(\zeta_i - \zeta_j\chi^j\chi^i) + \frac{1}{\beta}\epsilon_{ijk}(E^{bi}A_{bj} + \zeta_i\chi^j)A_{ak}\right] \ . \tag{55}$$

We split the gauge constraints into their boost and rotational components

$$\mathcal{G}^i_{boost} = \mathcal{G}^{0i} = \partial_a \left(E^{ai} - \frac{1}{\beta}\epsilon_{ijk}\chi^j E^{ak} \right) - E^{a[i}\chi^{j]}A_{aj} - \zeta_i + \zeta_j\chi^j\chi^i \tag{56}$$

$$\mathcal{G}^i_{rot} = \frac{1}{2}\epsilon_{ijk}\mathcal{G}^{jk} = -\partial_a \left(\epsilon_{ijk}\chi^j E^{ak} + \frac{1}{\beta}E^{ai} \right) + \epsilon_{ijk}(A_{aj}E^{ak} - \zeta_j\chi^k) \tag{57}$$

In order to write the scalar constraint in terms of the new variables the solution to $\mathcal{D}_{ab}$ is required, which is

$$M^{ij} = (f_{kk} - f_{kl}\chi^k\chi^l)\delta^{ij} - (f_{kk} + f_{kl}\chi^k\chi^l)\chi^i\chi^j - f_{ij} - f_{ji} + (f_{ik}\chi^j + f_{jk}\chi^i + f_{ki}\chi^j + f_{kj}\chi^i)\chi^k, \tag{58}$$

with

$$f_{ij} = \epsilon_{ikl}E^{ak}\left[(1+\beta^{-2})E_{bj}\partial_a E^{bl} + \chi^l A_{aj}\right] + \beta^{-1}(E^{bk}A_{bk}\delta_{ij} - A_{bi}E^{bj} - \zeta_i\chi^j) \ . \tag{59}$$

Here E_{ai} stands for the inverse of E^{ai}. The scalar constraint reads

$$\begin{aligned}\mathcal{H} = {} & E^{ai}\chi^i\mathcal{H}_a + (\chi^k\chi^k - 1)\left(E^{ai}\partial_a\zeta_i + \frac{1}{2}\zeta_i E^{ai}E^{bj}\partial_a E_{bj} \right) + \frac{\beta^2}{1+\beta^2}\left\{ (\chi^k\chi^k - 1)\left[\frac{3}{4}\zeta_i\zeta_i - \frac{3}{4}(\zeta_i\chi^i)^2 - \right.\right. \\ & -\zeta_i\chi^i A_{aj}E^{aj} - \frac{1}{2}E^{ai}E^{bj}A_{a[i|}A_{b|j]} - + \frac{1}{\beta}\epsilon_{ijl}\zeta_i A_{aj}E^{al} - \frac{1}{4}(f_{ij}+f_{ji})(f_{il}+f_{li})\chi^j\chi^l + \\ & \left.\left. + \frac{1}{2}f_{ij}\chi^i\chi^j(f_{ll} + f_{lm}\chi^l\chi^m) \right] - \frac{1}{4}\left[(f_{ii})^2 + (f_{ij}\chi^i\chi^j)^2\right] + \frac{1}{2}f_{ij}f_{ij} \right\} \ , \end{aligned} \tag{60}$$

In this formula we dropped terms proportional to the gauge constraints, as we did in (55). Therefore we ended up with a system described by 12 pairs of canonical variables (A_{ai}, E^{ai}) and (χ^i, ζ_i) subject to 10 first class constraints, $\mathcal{G}^i_{boost}$, $\mathcal{G}^i_{rot}$, $\mathcal{H}_a$ and $\mathcal{H}$. This is in accordance with the known result that gravity without matter presents 2 degrees of freedom per space point and that our theory is diffeomorphism invariant and possesses a further internal gauge symmetry, summing up to a 10 parameter family of symmetry transformations.

We followed this approach of introducing $\mathcal{C}^{ab}$ as primary constraints, generating secondary constraints and solving the resulting second class pairs, in order to keep in line with most of the literature in the area. But one can also insert (43)-(44) and (53)-(54) directly in the Lagrangian. Then expression (48) for the symplectic form follows and the field M_{ij} shows up only in the scalar constraint, in a quadratic form. Variation of the action with respect to M_{ij} leads to the solution. (58)-(59), rendering the two methods equivalent (In fact there is another solution, the vanishing of the lapse function $N = 0$, which we discard because it describes a 3 degree of freedom system of degenerate metrics. This same solution is also obtained using the method that we followed in this paper [10].)

The constraint analysis ends here. The system presents its full gauge symmetry and it is not yet in the form given by Barbero [1]. In order to do so one must fix the gauge freedom for boost transformations. We choose the time gauge

$$\chi^i = 0 \ , \tag{61}$$

and solve $\mathcal{G}^i_{boost}$ to obtain its canonical pair ζ_i,

$$\zeta_i = \partial_a E^{ai} \ . \tag{62}$$

Plugging these expressions into equations (55), (56), (57) and (60)-(59) one arrives at Barbero's form of gravity. In the next section we make a shortcut to this derivation.

V. GAUGE FIXING

One can safely fix the time gauge after all secondary constraints have been derived, and it turns out to be simpler to set the gauge fixing condition (61) before solving the second class constraints. Therefore we skip the last section and restart this section from the end of section III.

The gauge fixing condition (61) together with the constraints $\mathcal{C}^{ab}$ can be written in the form

$$\pi^{aij} = 0 \ , \tag{63}$$

and only the time-space components of the β-dependent connection show up in the symplectic form,

$$\frac{1}{2}\pi^{aIJ}\dot{\overset{(\beta)}{\omega}}_{aIJ} = E^{ai}\dot{A}_{ai} \ , \tag{64}$$

where

$$\overset{(\beta)}{\omega}_{a0i} = A_{ai} \ . \tag{65}$$

It is convenient to write the rotational part of the β-dependent connection in terms of A_{ai} and of the rotational part of the β-independent connection

$$\overset{(\beta)}{\omega}_{aij} = \epsilon_{ijk}\left[(1+\beta^{-2})\Gamma_{ak} + \alpha A_{ak}\right] \ . \tag{66}$$

with

$$\Gamma_{ai} = \frac{1}{2}\epsilon_{ijk}\omega_{ajk} \ . \tag{67}$$

The gauge constraints and $\mathcal{D}_{ab}$ become, using (63) and (65)-(66),

$$\mathcal{G}^i_{boost} = \partial_a E^{ai} + \epsilon_{ijk}\left[(1+\beta^{-2})\Gamma_{aj} + \beta^{-1}A_{aj}\right]E^{ak} \tag{68}$$

$$\mathcal{G}^i_{rot} = -\beta^{-1}\partial_a E^{ai} + \epsilon_{ijk}A_{aj}E^{ak} \tag{69}$$

$$\mathcal{D}^{ab} = \epsilon_{ijk}E^{ci}\left[(\partial_c E^{aj} + \epsilon_{jmn}\Gamma_{cm}E^{an})E^{bk} + (\partial_c E^{bj} + \epsilon_{jmn}\Gamma_{cm}E^{bn})E^{ak}\right] \ . \tag{70}$$

The constraints $\mathcal{D}^{ab}$ together with the following combination of the gauge constraints,

$$\frac{\beta}{1+\beta^2}(\beta\mathcal{G}^i_{boost} - \mathcal{G}^i_{rot}) = \partial_a E^{ai} + \epsilon_{ijk}\Gamma_{aj}E^{ak} \ , \tag{71}$$

can be solved for the variables Γ_{ai}. Equations (70)-(71) are equivalent to

$$D_a E^{bi} = \partial_a E^{bi} + \tilde{\Gamma}_{ac}{}^{b}E^{ci} - \tilde{\Gamma}_{ca}{}^{c}E^{bi} + \epsilon_{ijk}\Gamma_{aj}E^{bk} = 0 \ , \tag{72}$$

where $\tilde{\Gamma}_{ab}{}^{c}$ is the Riemannian connection constructed from the spatial metric (39)

$$q^{ab} = EE^{ai}E^{bi} \ , \tag{73}$$

with $E = 1/\det(E^{ai})$. Therefore Γ_{ai}, the rotational part of the β-independent spin-connection, is nothing but the spin-connection which annihilates the covariant derivative of the densitized tetrad E^{ai}. That is, with E_{ai} the inverse of E^{ai}, we may write the constraints (70)-(71) in the form

$$\Gamma_{ai} = \frac{1}{2}\epsilon_{ijk}E^{bj}\left(\partial_{[b}E_{a]k} + E_{a[l}E^{c}{}_{k]}\partial_b E_{cl}\right) \ , \tag{74}$$

which, considering the expression (67) for Γ_{ai} in terms of the original variables ${}^{(\beta)}\omega_{a0i}$ and ${}^{(\beta)}\omega_{aij}$, clearly form second class pairs with (63). In this way we have proven that the gauge fixing condition chosen is independent of the remaining constraints, that it forms second class pairs with the boosts, and that it does not destroy the second class relation between $\mathcal{C}^{ab}$ and $\mathcal{D}^{ab}$.

The remaining constraints are (we drop the label of the rotational gauge constraint)

$$\mathcal{G}^i = -\beta^{-1}D_a E^{ai} \tag{75}$$

$$\mathcal{H}_a = E^{bi}F_{abi} + (\beta A_{ai} + \Gamma_{ai})(\mathcal{G}^i_{rot} + \beta^{-1}\mathcal{G}^i_{boost}) \tag{76}$$

$$\mathcal{H} = -\beta^{-1}\frac{1}{2}\epsilon_{ijk}E^{ai}E^{bj}\left[F_{abk} + (\beta+\beta^{-1})R_{abk}\right] \ , \tag{77}$$

where

$$D_a E^{ai} = \partial_a E^{ai} - \beta\epsilon_{ijk}A_{aj}E^{ak} \tag{78}$$

and F_{abi} and R_{abi} stand for the curvature of A_{ai} and Γ_{ai} respectively,

$$F_{abi} = \partial_{[a}A_{b]i} - \beta\epsilon_{ijk}A_{aj}A_{bk} \tag{79}$$

$$R_{abi} = \partial_{[a}\Gamma_{b]i} + \epsilon_{ijk}\Gamma_{aj}\Gamma_{bk}\ . \tag{80}$$

We ended up with the pairs of canonical variables (A_{ai}, E^{ai}) subject to the constraints (75)-(76)-(77), which is Barbero's theory with coupling constant β [1].

Consistency can be checked by letting $\beta \to \infty$ and recovering the usual formulation of General relativity with tetrads [10], and setting $\alpha = \pm i$ to obtain Ashtekar's Hamiltonian [3]. We end this paper with a remark on Holst's calculation [5]. There the author fixes the gauge (61) before performing the Hamiltonian analysis. While this is a questionable method for a general gauge transformation, in this case the gauge fixing required the use of the gauge parameter and no time derivatives of it. It does not impose any conditions on the Lagrange multipliers and it corresponds to a so-called canonical gauge. Such a gauge fixing can be done directly in the Lagrangian without affecting locally the theory.

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Note added: After the completion of this work, a paper by S.Alexandrov [11] appeared concerning the same problem but with a somewhat different approach. To facilitate comparison I have adapted my notation a little.

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[1] J.Barbero, Phys.Rev.D 51 (1995) 5507.
[2] C.Rovelli, Living reviews, gr-qc 9710008.
[3] A.Ashtekar: New Perspectives in Canonical Gravity, Bibliopolis, Napoli 1988
[4] G.Immirzi, Class.Quant.Grav. 14 (1997) L177.
[5] S.Holst, Phys.Rev.D 53 (1996) 5966.
[6] C.Rovelli and T.Thiemann, Phys.Rev.D 57 (1998) 1009.
[7] A.Ashtekar, J.Baez and K.Krasnov, gr-qc/0005126.
[8] T.Jacobson and L.Smolin, Class.Quant.Grav. 5 (1988) 583
[9] J.Samuel, Pramana J.Phys. 28 (1987) L429.
[10] P.Peldan, Class.Quant.Grav. 11 (1994) 1087.
[11] S.Alexandrov, gr-qc/0005085.